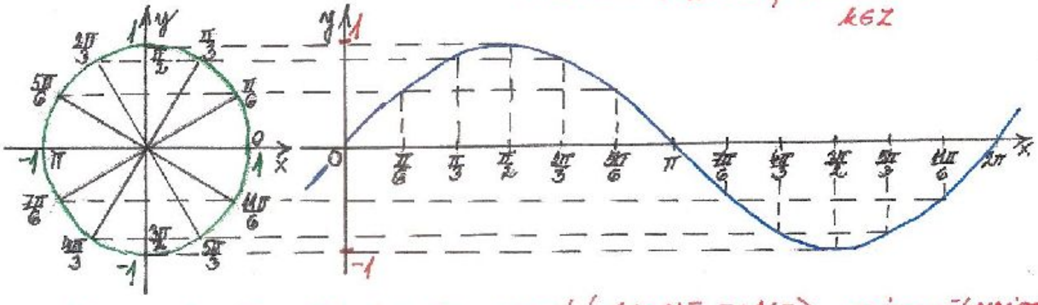
5. Grafy goniometrických funkcí

- pro sestrojení grafů využijeme jednotkové kružnice
- pro grafy $y = \cos x$ a $y = \operatorname{tg} x$ soustavu souřadnic otočíme o $90°$ $(\frac{\pi}{2})$ pro snazší odečítání hodnot

$f: y = \sin x$ — $D = R$, — $H = \langle -1, 1 \rangle$ — klesající $\langle \frac{\pi}{2} + 2k\pi; \frac{3\pi}{2} + 2k\pi \rangle$ — nejmenší perioda 2π

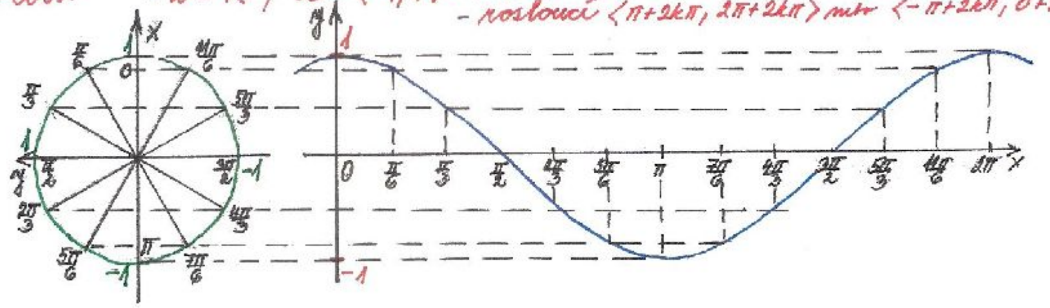
— rostoucí $\langle -\frac{\pi}{2} + 2k\pi; \frac{\pi}{2} + 2k\pi \rangle$

$k \in Z$

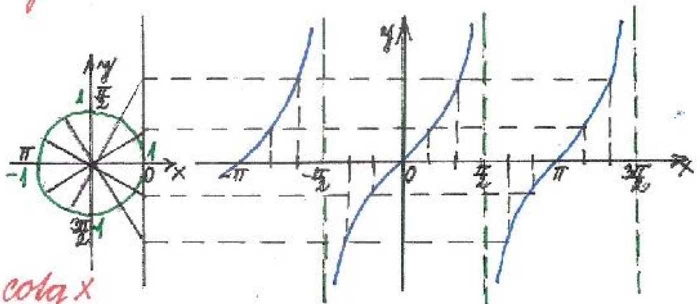


$f: y = \cos x$ — $D = R$, — $H = \langle -1, 1 \rangle$ — klesající $\langle 0 + 2k\pi, \pi + 2k\pi \rangle$ — nejmenší perioda 2π

— rostoucí $\langle \pi + 2k\pi, 2\pi + 2k\pi \rangle$ nebo $\langle -\pi + 2k\pi, 0 + 2k\pi \rangle$ $k \in Z$

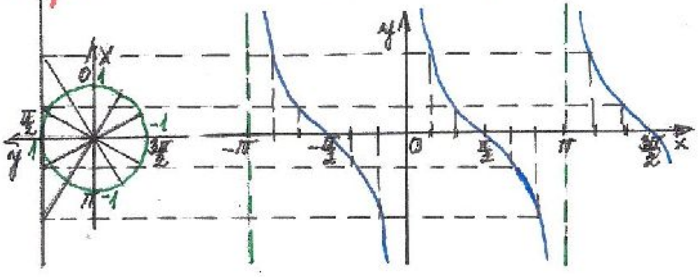


$f: y = \operatorname{tg} x$



- nedefinována pro LICHÉ NÁSOBKY $\frac{\pi}{2}$
  tj. $(2k+1)\frac{\pi}{2}$ $k \in Z$
  $D = R \setminus \bigcup_{k \in Z} \{(2k+1)\frac{\pi}{2}\}$
- obor hodnot
  $H = R$
- rostoucí $(-\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi)$
- nejmenší perioda π, $k \in Z$

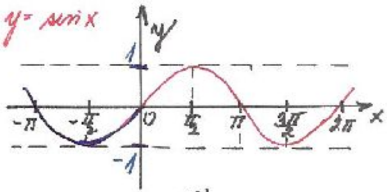
$f: y = \operatorname{cotg} x$

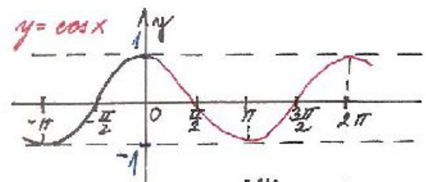


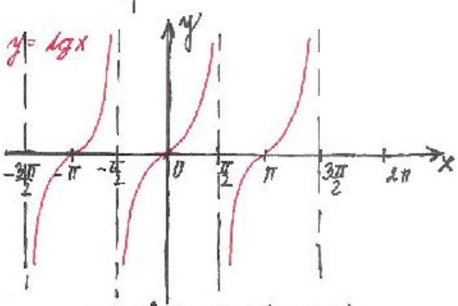
- nedefinována pro NÁSOBKY π
  tj. $k\pi$ $k \in Z$
  $D = R \setminus \bigcup_{k \in Z} \{k\pi\}$
- obor hodnot
  $H = R$
- klesající $(-\pi + k\pi, 0 + k\pi)$
  nebo $(0 + k\pi, \pi + k\pi)$
- nejmenší perioda π, $k \in Z$

TABULKA HODNOT GONIOMETRICKÝCH FUNKCÍ

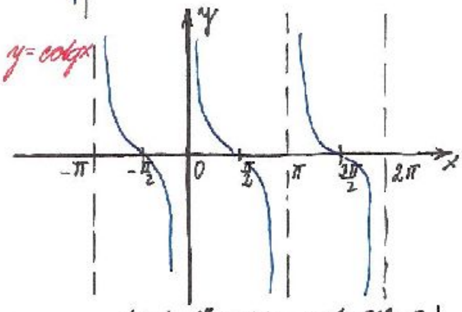
	1. kv. $0°$ 0	$30°$ $\frac{\pi}{6}$	$45°$ $\frac{\pi}{4}$	$60°$ $\frac{\pi}{3}$	$90°$ $\frac{\pi}{2}$	$180°$ π	$270°$ $\frac{3\pi}{2}$	$360°$ 2π	$-90°$ $-\frac{\pi}{2}$	$-180°$ $-\pi$
$\sin x$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0	-1	0
$\cos x$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0	1	0	-1
$\operatorname{tg} x$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	nedef.	0	nedef.	0	nedef.	0
$\operatorname{cotg} x$	nedef.	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0	nedef.	0	nedef.	0	nedef.







($\operatorname{tg} 0° = 0$, $\operatorname{tg} 90°$ nedef.)
=> graf, perioda $180°$



($\operatorname{cotg} 0°$ nedef., $\operatorname{cotg} 90° = 0$)
=> graf, perioda $180°$

Příklady

① Sestrojte grafy funkcí

$f: y = \sin x$

$H_f = \langle -1, 1 \rangle$

TYP: $y = a \sin x$

$g: y = 2 \sin x$

[pro $x = \frac{\pi}{2}$ $(90°) \Rightarrow y = 2 \underbrace{\sin 90°}_{1}$
$y = 2$]

=> 2x větší hodnoty y než fce $y = \sin x$

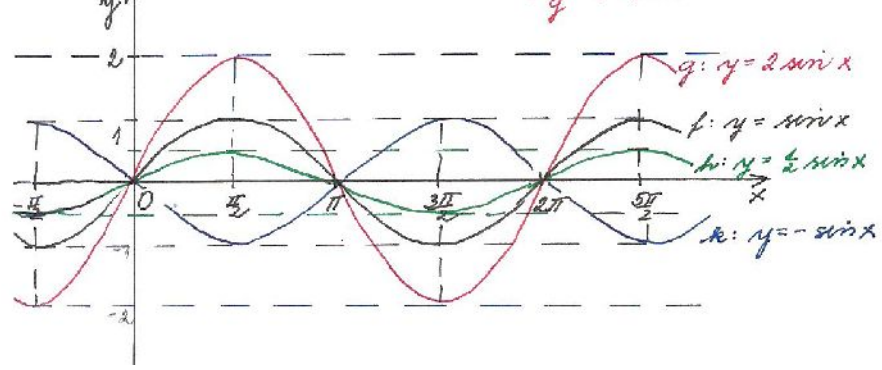
$H_g = \langle -2, 2 \rangle$

$h: y = \frac{1}{2} \sin x$ $\left(= \frac{\sin x}{2}\right)$

[$x = \frac{\pi}{2} \Rightarrow y = \frac{1}{2} \underbrace{\sin \frac{\pi}{2}}_{1} = \frac{1}{2}$]

=> poloviční hodnoty y než fce $y = \sin x$

$H_h = \langle -\frac{1}{2}, \frac{1}{2} \rangle$



$k: y = -\sin x$

[$x = \frac{\pi}{2} \Rightarrow y = -\underbrace{\sin \frac{\pi}{2}}_{1}$
$y = -1$]

=> hodnoty s opačným znaménkem, stejné velikosti než $y = \sin x$

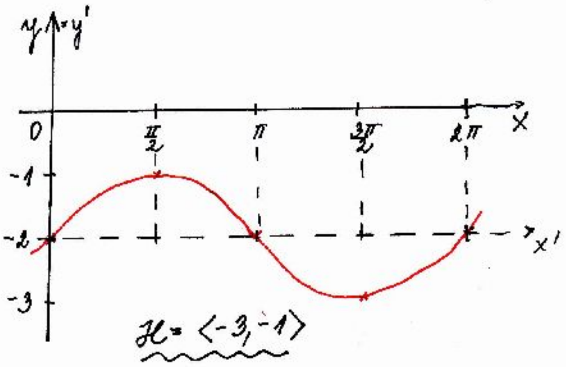
b) $f: y = \sin x - 2$ TYP: $y = \sin x - y_0$ $y = \cos x + 1$

$x_0 = 0$ $y_0 = -2$ $x_0 = 0$, $y_0 = 1$

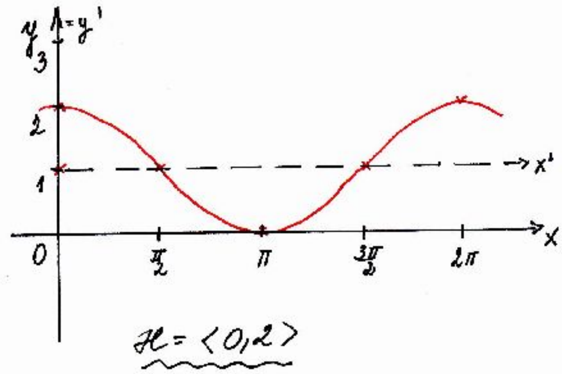
$[O'[0,-2]: y' = \sin x]$ $[O'[0,1]: y' = \cos x]$

funkční hodnoty posunuty o 2 ve směru záporné poloosy y

funkční hodnoty posunuty o 1 ve směru kladné poloosy y



$H = \langle -3, -1 \rangle$

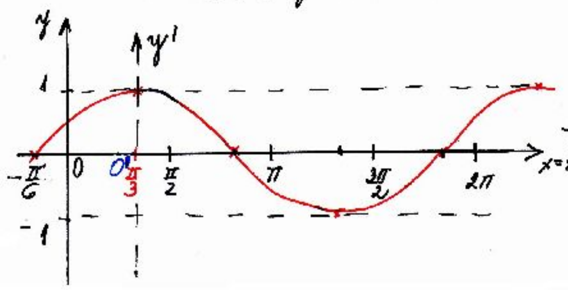


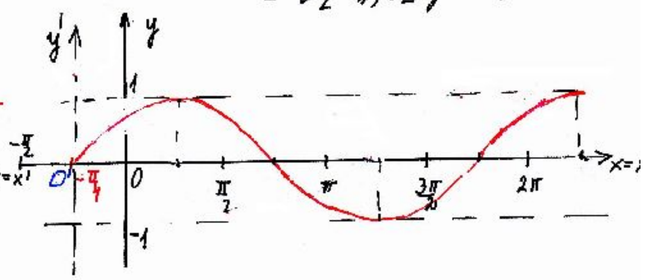
$H = \langle 0, 2 \rangle$

c) $f: y = \cos(x - \frac{\pi}{3})$ TYP: $y = \sin(x + x_0)$ $g: y = \sin(x + \frac{\pi}{4})$

$[x_0 = \frac{\pi}{3},\ y_0 = 0;\ O'[\frac{\pi}{3}, 0]\ y' = \cos x]$

$[x_0 = -\frac{\pi}{4},\ y_0 = 0;\ O'[-\frac{\pi}{4}, 0]\ y' = \sin x]$





d) $f: y = \sin 2x$ $[x_0 = 0,\ y_0 = 0\ O'[0,0]]$ TYP: $y = \sin bx$ $g: y = \cos \frac{x}{2}$ $[x_0 = 0,\ y_0 = 0\ O'[0,0]]$

! PERIODA 2X MENŠÍ než $y = \sin x$
tj. perioda $k\pi$ … „ZHUŠTĚNÍ"

$\sin 2x = \sin(2x + 2k\pi)$ $k \in Z$
$= \sin 2(x + k\pi)$

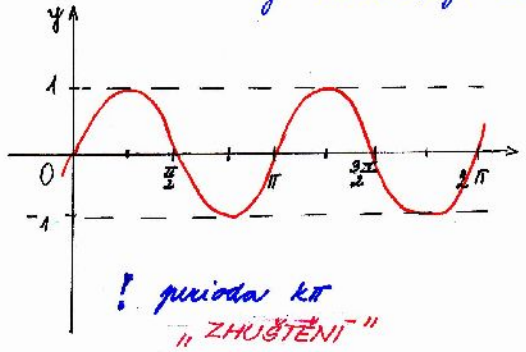
nejmenší perioda π
perioda $k\pi$, $k \in Z$

! $\cos \frac{1}{2} x$
PERIODA 2X VĚTŠÍ než $y = \sin x$
tj. perioda $2 \cdot 2k\pi = 4k\pi$

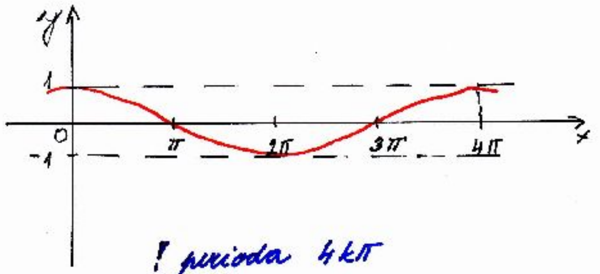
$\cos \frac{x}{2} = \cos(\frac{x}{2} + 2k\pi)$
$\cos \frac{x}{2} = \cos \frac{1}{2}(x + 4k\pi)$

nejmenší perioda 4π
perioda $4k\pi$, $k \in Z$

$y = \sin bx$, $y = \cos bx \Rightarrow$ nejmenší perioda $p = \frac{2\pi}{|b|}$

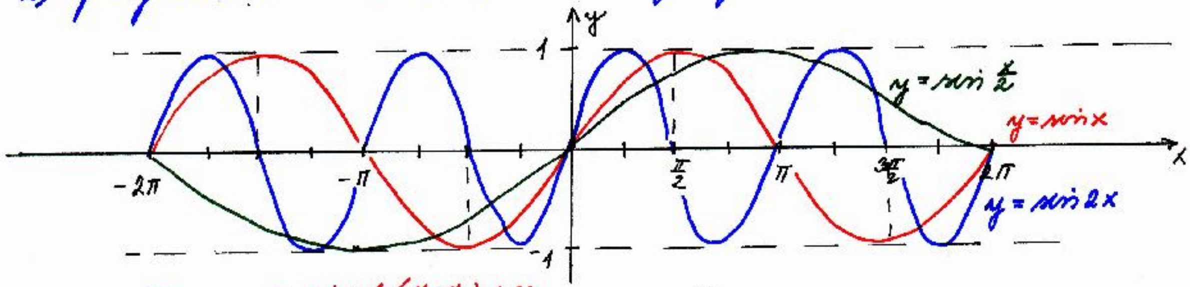


! perioda $k\pi$
„ZHUŠTĚNÍ"
nejmenší perioda π



! perioda $4k\pi$
„ZŘEDĚNÍ"
nejmenší perioda 4π

e) $f: y = \sin 2x \quad D = \langle -2\pi, 2\pi \rangle \qquad g: y = \sin \frac{x}{2} \quad D = \langle -2\pi, 2\pi \rangle$



TYP: $y = a \sin b(x - x_0) + y_0$

f) $y = 2 \sin\left(2x - \frac{\pi}{2}\right) + 1 = 2 \sin 2\left(x - \frac{\pi}{4}\right) + 1$ (x_0, y_0)

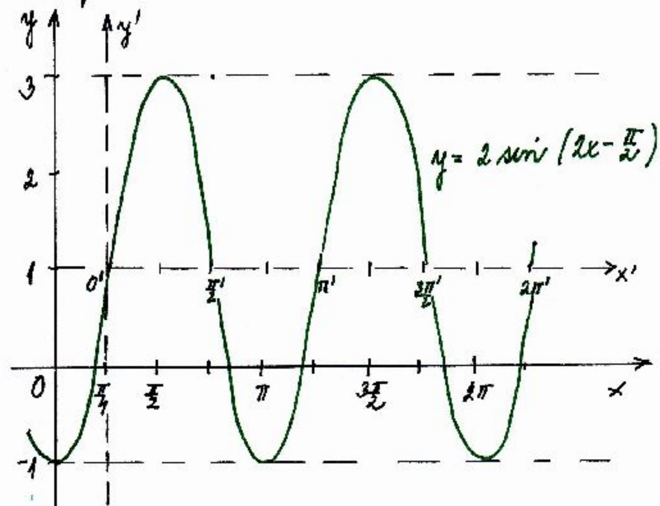
perioda 2x menší než $y = 2\sin x$, tj: perioda je $k\pi, k \in Z$

funkční hodnoty 2x větší než $y = \sin x$

$$\left[\begin{array}{ll} 2x_0 - \frac{\pi}{2} = 0 & y_0 = 1 \\ 2x_0 = \frac{\pi}{2} & \\ x_0 = \frac{\pi}{4} & O'\left[\frac{\pi}{4}, 1\right], \\ & f': y = 2 \sin 2x \end{array}\right]$$

obecně: $y = a \sin b(x - x_0) + y_0$

- hodnoty y a-krát VĚTŠÍ
- perioda b-krát MENŠÍ než $y = \sin x$
- posun ve směru osy x o x_0
- posun ve směru osy y o y_0



(v posunuté soustavě nakreslíme graf fce $y = 2\sin 2x$)

$D = R$

$H = \langle -1, 3 \rangle$

nejm. perioda π

g) $y = \frac{1}{2} \cos \frac{1}{2}(x + \pi) - 1$

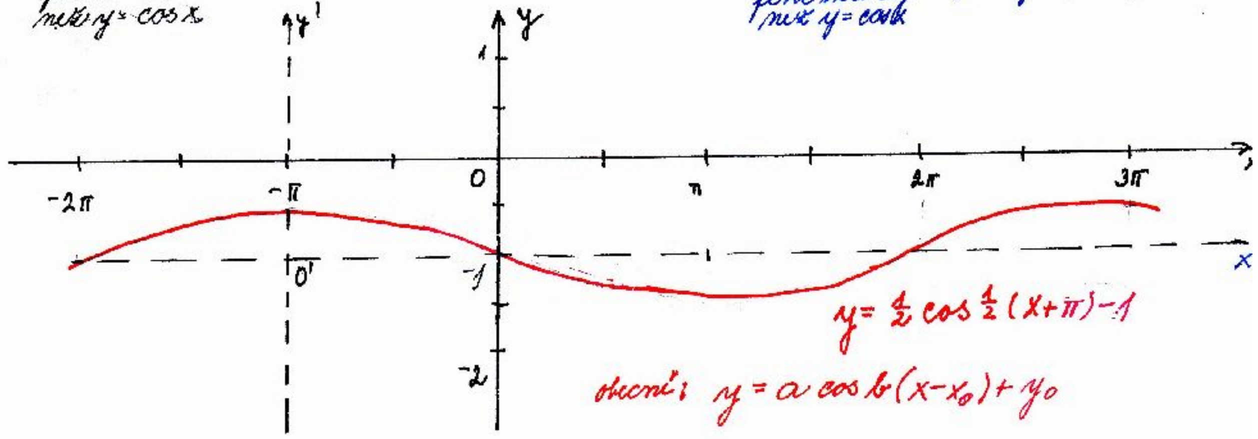
funkční hodnoty poloviční než $y = \cos x$

nejm. perioda 2x větší

$$\left[\begin{array}{l} x_0 = -\pi \\ y_0 = -1 \end{array}\right\} O'[-\pi, -1] \quad f': y = \frac{1}{2} \cos \frac{1}{2} x \Bigg]$$

poloviční fční hodnoty než $y = \cos x$

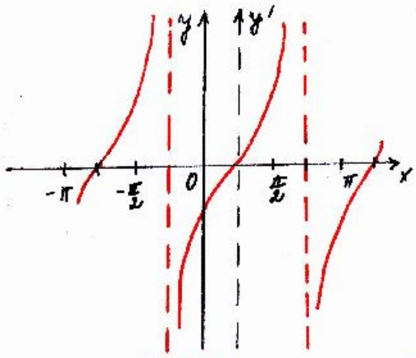
2x větší perioda než $y = \cos x$

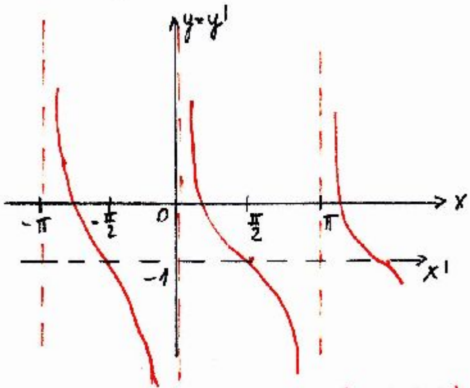


obecně: $y = a \cos b(x - x_0) + y_0$

h) $f: y = \mathrm{tg}\left(x - \frac{\pi}{4}\right)$ $\left[x_0 = \frac{\pi}{4}, y_0 = 0,\ O'\left[\frac{\pi}{4}, 0\right],\ y' = \mathrm{tg}\, x\right]$

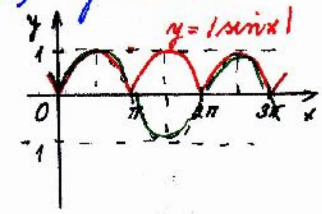
$g: y = \mathrm{cotg}\, x - 1$ $\left[x_0 = 0, y_0 = -1\quad O'[0,-1],\ y' = \mathrm{cotg}\, x\right]$





TYP: $y = |\sin x|$

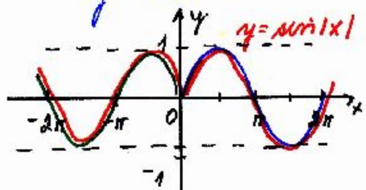
ch) $y = |\sin x|$



[hodnoty fce (*) $y = \sin x$ pod osou x sestr. v osové soum. podle osy x (*)]

TYP: $y = \sin|x|$

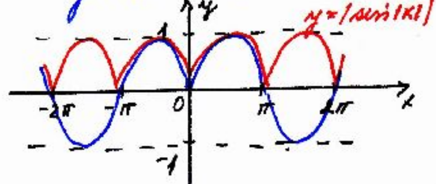
$y = \sin|x|$



[$y = |x|$ ... sudá fce (soum. podle osy y)
1. $y = \sin x$ v $\langle 0, +\infty)$ (*)
2. v $(-\infty, 0)$ graf soum. podle osy y (*)]

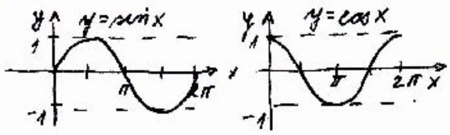
TYP: $y = |\sin|x||$

$y = |\sin|x||$



[1. graf fce $y = \sin|x|$
2. záp. hodnoty v osové soum. podle osy x]

(2) Vypočítej s využitím grafů gon. funkcí



DÚ! a) $a^2 \underbrace{\sin\frac{\pi}{2}}_{1} + b^2 \underbrace{\cos 0}_{1} + 2ab \underbrace{\cos \pi}_{-1} =$

$= a^2 + b^2 - 2ab = (a-b)^2$

b) $3 \underbrace{\cos\frac{\pi}{2}}_{0} - 4 \underbrace{\sin\frac{3}{2}\pi}_{-1} + 8 \underbrace{\mathrm{tg}\,\pi}_{0} = -4(-1) = 4$

c) $2 \underbrace{\cos\pi}_{-1} + 6 \underbrace{\mathrm{cotg}\frac{3\pi}{2}}_{0} - 5 \underbrace{\sin 2\pi}_{0} = -2 + 6\cdot 0 - 5\cdot 0 = -2$

d) $2 \underbrace{\mathrm{tg}\,0}_{0} + \underbrace{\sin\pi}_{0} - \underbrace{\cos\frac{3}{2}\pi}_{0} - \underbrace{\mathrm{cotg}\frac{\pi}{2}}_{-1} = 2\cdot 0 + 0 - 0 - (-1) = 1$

(3) Vypočítej

DÚ! a) $\mathrm{tg}\,30^\circ \cdot \mathrm{cotg}\,30^\circ - \sin 30^\circ \cdot \mathrm{tg}\,60^\circ = \frac{\sqrt{3}}{3}\cdot\sqrt{3} - \frac{1}{2}\cdot\sqrt{3} = 1 - \frac{\sqrt{3}}{2}$

b) $\mathrm{cotg}\frac{12\pi}{8} \cdot \mathrm{tg}\,11\pi \cdot \mathrm{cotg}\frac{19\pi}{3} \cdot \mathrm{tg}(-7\pi)$ $\left[\mathrm{cotg}\frac{12\pi}{8} = \mathrm{cotg}\frac{\pi}{2} =\right.$

$= 0 \cdot \mathrm{tg}\,\pi \cdot \mathrm{cotg}\frac{\pi}{3}(-\mathrm{tg}\,7\pi) = 0$

c) $\dfrac{\mathrm{tg}\left(-\frac{\pi}{4}\right)\cdot \mathrm{cotg}\left(-\frac{\pi}{4}\right)}{\sin\left(-\frac{3\pi}{2}\right)\cdot\cos(-4\pi)} = \dfrac{+\mathrm{tg}\frac{\pi}{4}\cdot\mathrm{cotg}\frac{\pi}{4}}{-\sin\frac{3\pi}{2}\cdot\cos 4\pi} = \dfrac{1\cdot 1}{-(-1)\cdot 1} = 1$